Inter-temporal Interactive Decisions and Dynamic Games

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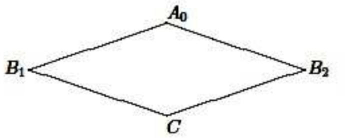
Hierarchical games (cooperative version)

Hierarchical systems with double subordination are called diamond-shaped. Control of a double subordination division C depends on control B_1 and control B_2.



One can envision a situation in which center B_1 represents the interests of an industry, while B_2 represents regional interests, including the issues of environment protection. A simple diamond-shaped system is an example of a hierarchical two-level decision-making system. At the upper level there is an administrative center which is in charge of material and labor resources. It brings an influence to bear upon activities of its two subordinate centers belonging to the next level. The decisions made by these centers determine an output of the enterprise standing at a lower level of the hierarchical system.

Hierarchical games (cooperative version)

We shall consider this decision-making process as a four-person game. Denote this game by Γ. Going to the game setting, we assume that at the first step Player A_0 moves and selects an element (strategy) $u = (u_1, u_2)$ from a certain set U, where U is a strategy set for Player A_0. The element $u \in U$ restricts the possibilities for players B_1 and B_2 to make their choices at the next step. In other words, the set of choices for Player B_1 is function of the parameter u_1 (denoted by $B_1(u_1)$). Similarly, the set of choices for Player B_2 is function of parameter u_2 (denoted by $B_2(u_2)$). Denote by $\omega_1 \in B_1(u_1)$ and $\omega_2 \in B_2(u_2)$ the elements of the sets of choices for players B_1 and B_2 respectively. The parameters ω_1 and ω_2 selected by players B_1 and B_2 specify restrictions on the set of choices for Player C at the third step of the game, i.e. this set turns out to be the function of parameters ω_1 and ω_2. Denote it by $C(\omega_1, \omega_2)$, and the elements of this set (production programs) by v.

Hierarchical games (cooperative version)

Suppose the payoffs of all players A_0, B_1, B_2, C depend only on the production program v selected by Player C and are respectively equal to $l_1(v), l_2(v), l_3(v), l_4(v)$, where $l_i(v) \geq 0$.
This hierarchical game can be represented as a noncooperative four-person game in normal form if the strategies for Player A_0 are taken to be the elements $u = (u_1, u_2) \in U$, while the strategies for players B_1, B_2 and C are taken to be the functions $\omega_1(u_1), \omega_2(u_2)$ and $v(\omega_1, \omega_2)$ with values in the sets $B_1(u_1), B_2(u_2), C(\omega_1, \omega_2)$, respectively, (the sets of such functions will be denoted by B_1, B_2, C) which set up a correspondence between every possible choice by the player (or the players) standing at a higher level and the choice made by this player. Setting

$$K_i(u, \omega_1(\cdot), \omega_2(\cdot), v(\cdot)) = l_i(v(\omega_1(u_1), \omega_2(u_2)), \;\; i = \overline{1,4}$$

we obtain the normal form of the game Γ

$$\Gamma = (U, B_1, B_2, C, K_1, K_2, K_3, K_4).$$

Multistage games with incomplete information

We considered multistage games with perfect information defined in terms of a finite tree graph $G = (X, F)$ in which each of the players exactly knows at his move the position or the tree node where he stays. That is why we were able to introduce the notion of player i's strategy as a single-valued function $u_i(x)$ defined on the set of personal positions X_i with its values in the set F_x. If, however, we wish to study a multistage game in which the players making their choices have no exact knowledge of positions in which they make their moves or may merely speculate that this position belongs to some subset A of personal positions X_i, then the realization of player's strategy as a function of position $x \in X_i$ turns out to be impossible. In this manner the wish to complicate the information structure of a game inevitably involves changes in the notion of a strategy. In order to provide exact formulations, we should first formalize the notion of information in the game. Here the notion of an information set plays an important role. This will be illustrated with some simple, already classical examples from texts on game theory.

Zero-sum game

Example 1. Player 1 selects at the first move a number from the set $\{1, 2\}$. The second move is made by Player 2. He is informed about Player 1's choice and selects a number from the set $\{1, 2\}$. The third move is again to be made by Player 1. He knows Player 2's choice, remembers his own choice and selects a number from the set $\{1, 2\}$. At this point the game terminates and Player 1 receives a payoff H (Player 2 receives a payoff $(-H)$, i.e. the game is zero-sum), where the function H is defined as follows:

$$H(1,1,1) = -3,\ H(2,1,1) = 4,$$

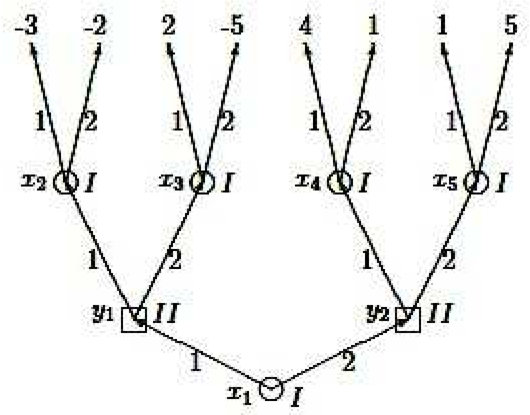
$$H(1,1,2) = -2,\ H(2,1,2) = 1,$$

$$H(1,2,1) = 2,\ H(2,2,1) = 1,$$

$$H(1,2,2) = -5,\ H(2,2,2) = 5,$$

The graph $G = (X, F)$ of the game is depicted in Figure. The circles in the graph represent positions in which Player 1 makes a move, whereas the blocks represent positions in which Player 2 makes a move.

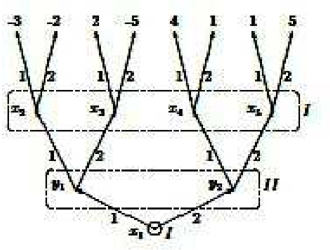
Zero-sum game



Zero-sum game

If the set X_1 is denoted by X, the set X_2 by Y and the elements of these sets by $x \in X$, $y \in Y$, respectively, then Player 1's strategy $u_1(\cdot)$ is given by the five-dimensional vector
$u_1(\cdot) = \{u_1(x_1), u_1(x_2), u_1(x_3), u_1(x_4), u_1(x_5)\}$ prescribing the choice of one of the two numbers $\{1, 2\}$ in each position of the set X. Similarly, Player 2's strategy $u_2(\cdot)$ is a two-dimensional vector
$u_2(\cdot) = \{u_2(y_1), u_2(y_2)\}$ prescribing the choice of one of the two numbers $\{1, 2\}$ in each of the positions of the set Y. Now, in this game Player 1 has 32 strategies and Player 2 has 4 strategies. The corresponding normal form of the game has a 32×4 matrix which has an equilibrium in pure strategies. It can be seen that the value of this game is 4. Player 1 has four optimal pure strategies: (2,1,1,1,2), (2,1,2,1,2), (2,2,1,1,2), (2,2,2,1,2). Player 2 has two optimal strategies: (1,1), (2,1).

Example 2. Player 1 chooses a number from the set $\{1, 2\}$ on the first move. The second move is made by Player 2 without being informed about Player 1's choice. Further, on the third move Player 1 chooses a number from the set $\{1, 2\}$ without knowing Player 2's choice and with no memory of his own choice at the first step.



Here the strategy of Player 1 consists of a pair of numbers (i, j), the i-th choice is at the first step, and j-th choice is at the third step; the strategy of Player 2 is a choice of number j at the second step of the game. Now, Player 1 has four strategies and Player 2 has two strategies. The game in normal form has a 4×2 matrix:

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} (1.1) \\ (1.2) \\ (2.1) \\ (2.2) \end{array} & \begin{bmatrix} -3 & 2 \\ -2 & -5 \\ 4 & 1 \\ 1 & 5 \end{bmatrix} \end{array}.$$

The value of the game is $19/7$, an optimal mixed strategy for Player 1 is $(0, 0, 4/7, 3/7)$, whereas an optimal strategy for Player 2 is $(4/7, 3/7)$. In this game the value is found to be the same as in Example 8, i.e. it turns out that the deterioration of information conditions for Player 2 did not improve the state of Player 1. This condition is random in nature and is accountable to special features of the payoff function.

Cooperative multistage games with perfect information

In what follows as basic model we shall consider the game in extensive form with perfect information.
Suppose that finite oriented treelike graph G with the root x_0 is given. For simplicity we shall use the following notations. Let x be some vertex (position). We denote by G(x) a subtree of G with root in x. We denote by $Z(x)$ immediate successors of x. As before the vertices y, directly following after x, are called alternatives in x ($y \in Z(x)$). The player who makes a decision in x (who selects the next alterative position in x), will be denoted by $i(x)$. The choice of player $i(x)$ in position x will be denoted by $\bar{x} \in Z(x)$.

Cooperative multistage games with perfect information

Definition 1

A game in extensive form with perfect information (see []) $\Gamma(x_0)$ is a graph tree $G(x_0)$, with the following additional properties:

1. The set of vertices (positions) is split up into $n+1$ subsets $X_1, X_2, \ldots, X_{n+1}$, which form a partition of the set of all vertices of the graph tree $G(x_0)$. The vertices (positions) $x \in X_i$ are called players i personal positions, $i = 1, \cdots, n$; vertices (positions) $x \in X_{n+1}$ are called terminal positions.
2. For each vertex $x \notin X_{n+1}$ and $y \in Z(x)$ define an arc (x,y) on the graph $G(x_0)$. On each arc (x, y) n real numbers (payoffs of players on this arc) $h_i(x, y)$, $i = 1, \ldots, n$, $h_i \geq 0$ are defined, and also terminal payoffs $g_i(x) \geq 0$, for $x \in X_{n+1}$, $i = 1, \ldots, n$.

Cooperative multistage games with perfect information

Definition 2

A strategy of player i is a mapping $U_i(\cdot)$, which associate to each position $x \in X_i$ a unique alternative $y \in Z(x)$.

Denote by $H_i(x; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function of player $i \in N$ in the subgame $\Gamma(x)$ starting from the position x.

$$H_i(x; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{k=0}^{l-1} h_i(x_k, x_{k+1}) + g_i(x_l), \quad h_i \geq 0, \quad g_i \geq 0$$

where $x_l \in X_{n+1}$ is the last vertex (position) in the path $\tilde{x} = (x_0, x_1, \dots, x_l)$ realized in subgame $\Gamma(x)$, and $x_0 = x$, when n-tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played.

Cooperative multistage games with perfect information

Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the n-tuple of strategies and the trajectory (path) $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$, $\bar{x}_l \in P_{n+1}$ such that

$$\max_{u_1(\cdot),\dots,u_n(\cdot)} \sum_{i=1}^{n} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) =$$

$$= \sum_{i=1}^{n} H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^{n} \left(\sum_{k=0}^{l-1} h_i(\bar{x}_k, \bar{x}_{k+1}) + g_i(\bar{x}_l) \right). \quad (1)$$

The path $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$ satisfying (1) we shall call "optimal cooperative trajectory".

Define in $\Gamma(x_0)$ characteristic function in a classical way

$$V(x_0; N) = \sum_{i=1}^{n} \left(\sum_{k=0}^{l-1} h_i(\bar{x}_k, \bar{x}_{k+1}) + g_i(\bar{x}_l) \right),$$

$$V(x_0; \emptyset) = 0,$$

$$V(x_0; S) = Val\Gamma_{S,N\setminus S}(x_0),$$

where $Val\Gamma_{S,N\setminus S}(x_0)$ is a value of zero-sum game played between coalition S acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to

$$\sum_{i\in S} H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot)).$$

Cooperative multistage games with perfect information

If the characteristic function is defined then we can define the set of imputations in the game $\Gamma(x_0)$

$$C(x_0) = \left\{ \xi = (\xi_1, \ldots, \xi_n) : \quad \xi_i \geq V(x_0; \{i\}), \quad \sum_{i \in N} \xi_i = V(x_0; N) \right\},$$

the core $M(x_0) \subset C(x_0)$

$$M(x_0) = \left\{ \xi = (\xi_1, \ldots, \xi_n) : \quad \sum_{i \in S} \xi_i \geq V(x_0; S), \quad S \subset N \right\} \subset C(x_0),$$

NM solution, Shapley value and other optimality principles of classical game theory. In what follows we shall denote by $M(x_0) \subset C(x_0)$ anyone of this optimality principles.

Cooperative multistage games with perfect information

Suppose at the beginning of the game players agree to use the optimality principle $M(x_0) \subset C(x_0)$ as the basis for the selection of the "optimal" imputation $\bar{\xi} \in M(x_0)$.
This means that playing cooperatively by choosing the strategy maximizing the common payoff each one of them is waiting to get the payoff $\bar{\xi}_i$ from the optimal imputation $\bar{\xi} \in M(x_0)$ after the end of the game (after the maximal common payoff $V(x_0; N)$ is really earned by the players).
But when the game Γ actually develops along the "optimal" trajectory $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_l)$ at each vertex $\bar{x}_k$ the players find themselves in the new multistage game with perfect information $\Gamma_{\bar{x}_k}$, $k = 0, \dots, l$, which is the subgame of the original game Γ starting from $\bar{x}_k$ with payoffs

$$H_i(\bar{x}_k; u_1(\cdot), \dots, u_n(\cdot)) = \sum_{j=k}^{l-1} h_i(x_j, x_{j+1}) + g_i(x_l), \quad i = 1, \dots, n.$$

It is important to mention that for the problem (1) the Bellman optimality principle holds and the part $\bar{x}^k = (\bar{x}_k, \ldots, \bar{x}_j, \ldots, \bar{x}_l)$ of the trajectory $\bar{x}$, starting from $\bar{x}_k$ maximizes the sum of the payoffs in the subgame $\Gamma_{\bar{x}_k}$, i.e.

$$\max_{x_k, \ldots, x_j, \ldots, x_l} \sum_{i=1}^{n} \left[\sum_{j=k}^{l-1} h_i(x_j, x_{j+1}) + g_i(x_l) \right] = \sum_{i=1}^{n} \left[\sum_{j=k}^{l-1} h_i(\bar{x}_j, \bar{x}_{j+1}) + g_i(\bar{x}_l) \right],$$

which means that the trajectory $\bar{x}^k = (\bar{x}_k, \ldots, \bar{x}_j, \ldots, \bar{x}_l)$ is also "optimal" in the subgame $\Gamma_{\bar{x}_k}$.

Cooperative multistage games with perfect information

Before entering the subgame $\Gamma_{\bar{x}_k}$ each of the players i have already earned the amount

$$H_i^{\bar{x}_k} = \sum_{j=0}^{k-1} h_i(\bar{x}_j, \bar{x}_{j+1}).$$

At the same time at the beginning of the game $\Gamma = \Gamma(x_0)$ the player i was oriented to get the payoff $\bar{\xi}_i$ – the ith component of the "optimal" imputation $\bar{\xi} \in M(x_0) \subset C(x_0)$. From this it follows that in the subgame $\Gamma_{\bar{x}_k}$ he is expected to get the payoff equal to

$$\bar{\xi}_i - H_i^{\bar{x}_k} = \bar{\xi}_i^{\bar{x}_k}, \quad i = 1, \dots, n$$

and then the question arises whether the new vector $\bar{\xi}^{\bar{x}_k} = (\bar{\xi}_1^{\bar{x}_k}, \dots, \bar{\xi}_i^{\bar{x}_k}, \dots, \bar{\xi}_n^{\bar{x}_k})$ remains to be optimal in the same sense in the subgame $\Gamma_{\bar{x}_k}$ as the vector $\bar{\xi}$ was in the game $\Gamma(\bar{x}_0)$. If this will not be the case, it will mean that the players in the subgame $\Gamma_{\bar{x}_k}$ will not orient themselves on the same optimality principle as in the game $\Gamma(\bar{x}_0)$ which may enforce them to go out from the cooperation by changing the chosen cooperative strategies $\bar{u}_i(\cdot)$, $i = 1, \dots, n$ and thus changing the optimal trajectory $\bar{x}$ in the subgame $\Gamma(\bar{x}_k)$. Try now formalize this reasoning.

Introduce in the subgame $\Gamma(\bar{x}_k)$, $k = 1, \ldots, l$, the characteristic function $V(\bar{x}_k; S)$, $S \subset N$ in the same manner as it was done in the game $\Gamma = \Gamma(x_0)$. Based on the characteristic function $V(\bar{x}_k; S)$ we can introduce the set of imputations

$$C(\bar{x}_k) = \left\{ \xi = (\xi_1, \ldots, \xi_n) : \quad \xi_i \geq V(\bar{x}_k; \{i\}), \quad \sum_{i \in N} \xi_i = V(\bar{x}_k; N) \right\},$$

the core $M(\bar{x}_k) \subset C(\bar{x}_k)$

$$M(\bar{x}_k) = \left\{ \xi = (\xi_1, \ldots, \xi_n) : \quad \sum_{i \in S} \xi_i \geq V(\bar{x}_k; S), \quad S \subset N \right\} \subset C(\bar{x}_k),$$

NM solution, Shapley value and other optimality principles of classical game theory. Denote by $M(\bar{x}_k) \subset C(\bar{x}_k)$ the optimality principle $M \subset C$ (which was selected by players in the game $\Gamma(x_0)$) considered in the subgame $\Gamma(\bar{x}_k)$.

Cooperative multistage games with perfect information

If we suppose that the players in the game $\Gamma(x_0)$ when moving along the optimal trajectory $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_l)$ follow the same ideology of optimal behaviour then the vector $\bar{\xi}^{\bar{x}_k} = \bar{\xi} - H^{\bar{x}_k}$ must belong to the set $M(\bar{x}_k)$ – the corresponding optimality principle in the cooperative game $\Gamma(\bar{x}_k)$, $k = 0, \ldots, l$.

It is clearly seen that it is very difficult to find games and corresponding optimality principles for which this condition is satisfied. Try to illustrate this on the following example.

Suppose that in the game Γ $\quad h_i(x_k, x_{k+1}) = 0, k = 0, \ldots, l-1$, $g_i(x_l) \neq 0$ (the game Γ is the game with terminal payoff). Then the last condition would mean that

$$\bar{\xi} = \bar{\xi}^{\bar{x}_k} \in M(\bar{x}_k), \quad k = 0, \ldots, l,$$

which gives us

$$\bar{\xi} \in \bigcap_{k=0}^{l} M(\bar{x}_k). \tag{2}$$

Cooperative multistage games with perfect information

For $k = l$ we shall have that

$$\bar{\xi} \in M(\bar{x}_l).$$

But $M(\bar{x}_l) = C(\bar{x}_l) = \{g_i(\bar{x}_l)\}$. And this condition have to be valid for all imputations of the set $M(\bar{x}_0)$ and for all optimality principles $M(x_0) \subset C(x_0)$, which means that in the cooperative game with terminal payoffs the only reasonable optimality principle will be

$$\bar{\xi} = \{g_i(\bar{x}_l)\},$$

the payoff vector obtained at the end point of the cooperative trajectory in the game $\Gamma(x_0)$. At the same time the simplest examples show that the intersection (2) except the "dummy" cases, is void for the games with terminal payoffs. How to overcome this difficulty. The plausible way of finding the outcome is to introduce a special rule of payments (stage salary) on each stage of the game in such a way that the payments on each stage will not exceed the common amount earned by the players on this stage and the payments received by the players starting from the stage k(in the subgame $\Gamma(\bar{x}_k)$) will belong to the same optimality principle as the imputation $\bar{\xi}$ on which players agree in the game $\Gamma(x_0)$ at the beginning of the game. Whether it is possible or not we shall consider now.

Cooperative multistage games with perfect information

Introduce the notion of the imputation distribution procedure (IDP).

Definition 3

Suppose that $\xi = \{\xi_1, \dots, \xi_i, \dots, \xi_n\} \in M(x_0)$. Any matrix $\beta = \{\beta_{ik}\}$, $i = 1, \dots, n$, $k = 0, \dots, n$ such that

$$\xi_i = \sum_{k=0}^{l} \beta_{ik}, \tag{3}$$

is called the imputation distribution procedure (IDP).

Denote $\beta_k = (\beta_{1k}, \dots, \beta_{nk})$, $\beta(k) = \sum_{m=0}^{k-1} \beta_m$. The interpretation of IDP β is: β_{ik} is the payment to player i on the stage k of the game Γ_{x_0}, i.e. on the first stage of the subgame $\Gamma(\bar{x}_k)$. From the definition (3) it follows that in the game $\Gamma(x_0)$ each player i gets the amount ξ_i, $i = 1, \dots, n$, which he expects to get as the ith component of the optimal imputation $\xi_i \in M(x_0)$ in the game $\Gamma(x_0)$. The interpretation of $\beta_i(k)$ is: $\beta_i(k)$ is the amount received by player i on the first k stages of the game Γ_{x_0}.

Cooperative multistage games with perfect information

Definition 4

The optimality principle $M(x_0)$ is called time-consistent if for every $\xi \in M(x_0)$ there exists IDP β such that

$$\xi^k = \xi - \beta(k) \in M(\bar{x}_k), \quad k = 0, 1, \dots, l. \tag{4}$$

Definition 5

The optimality principle $M(x_0)$ is called strongly time-consistent if for every $\xi \in M(x_0)$ there exists IDP β such that

$$\beta(k) \oplus M(\bar{x}_k) \subset M(x_0), \quad k = 0, 1, \dots, l.$$

Here $a \oplus A = \{a + a' : \ a' \in A, a \in R^n, A \subset R^n\}$.

The time-consistency of the optimality principle $M(x_0)$ implies that for each imputation $\xi \in M$ there exits such IDP β that if the payments on each arc $(\bar{x}_k, \bar{x}_{k+1})$ on the optimal trajectory $\bar{x}$ will be made to the players according to IDP β, in every subgame $\Gamma(\bar{x}_k)$ the players may expect to receive the payments $\bar{\xi}^k$ which are optimal in the subgame $\Gamma(\bar{x}_k)$ in the same sense as it was in the game $\Gamma(x_0)$.

Cooperative multistage games with perfect information

The strongly time-consistency means that if the payments are made according to IDP β then after earning on the stage k amount $\beta(k)$ the players (if they are oriented in the subgame $\Gamma(\bar{x}_k)$ on the same optimality principle as in $\Gamma(x_0)$) start with reconsidering of the imputation in this subgame (but optimal) they will get as a result in the game $\Gamma(x_0)$ the payments according to some imputation, optimal in the previous sense, i.e. the imputation belonging to the set $M(x_0)$.
For any optimality principle $M(x_0) \subset C(x_0)$ and for every $\bar{\xi} \in M(x_0)$ we can define β_{ik} by the following formulas

$$\bar{\xi}_i^{\bar{x}_k} - \bar{\xi}_i^{\bar{x}_{k+1}} = \beta_{ik}, \quad i = 1, \ldots, n, \quad k = 0, \ldots, l-1,$$
$$\bar{\xi}_i^{\bar{x}_l} = \beta_{il}. \tag{5}$$

From the definition it follows that

$$\sum_{k=0}^{l} \beta_{ik} = \sum_{k=0}^{l-1} (\bar{\xi}_i^{\bar{x}_k} - \bar{\xi}_i^{\bar{x}_{k+1}}) + \bar{\xi}_i^{\bar{x}_l} = \bar{\xi}_i^{\bar{x}_0} = \bar{\xi}_i.$$

And at the same time

$$\bar{\xi} - \beta(k) = \bar{\xi}^{\bar{x}_k} \in M(\bar{x}_k), k = 0, \ldots, l.$$

The last inclusion would mean the time consistency $M(x_0)$.
Unfortunately the elements β_{ik} may take in many cases negative values, which may stimulate questions about the use of this payment mechanism in real life situations. Because this means that players in some cases have to pay to support time–consistency. We understand that this argument can be waved since the total amount the player gets in the game is equal to the component ξ_i of the optimal imputation, and he can borrow the money to cover the side payment β_{ik} on stage k.
But we have another approach which enables us to use only nonnegative IDP's, and get us result not only time-consistent, but strongly time-consistent solution. For this reason some integral transformation of characteristic function is needed.

Time inconsistency of the Shapley Value

Example 3. . Consider the cooperative version of the game on the Figure, in the case when there are only 3 players. The following coalitions are possible $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$, $\{3\}$. The characteristic function has the form $v(\{1,2,3\}) = 6$, $v(\{1,2\}) = 2$, $v(\{1,3\}) = 2$, $v(\{2,3\}) = 2$, $v(\{1\}) = 1$, $v(\{2\}) = \frac{1}{2}$, $v(\{3\}) = \frac{1}{2}$. Computing the Shapley Value we get

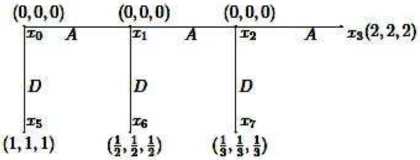
$$Sh(x_0) : Sh_1 = \frac{26}{12},\ Sh_2 = \frac{23}{12},\ Sh_3 = \frac{23}{12}.$$

Suppose the game develops along the optimal cooperative trajectory, which corresponds to the choices (A, A, A), and coincides with the path $\bar{x} = (x_0, x_1, x_2, x_3)$. As we have seen $v(x_0; \{1,2,3\}) = 6$, $v(x_0; \{1,2\})= v(x_0; \{1,3\}) =v(x_0; \{2,3\}) = 2$, $v(x_0; \{1\}) = 1$, $v(x_0; \{2\}) =v(x_0; \{3\}) = \frac{1}{2}$. Consider now the subgame starting on cooperative trajectory from vertex $\bar{x}_1$. It can be easily seen that $v(\bar{x}_1; \{1,2,3\}) = 6$, $v(\bar{x}_1; \{1,2\}) = 1$, $v(\bar{x}_1; \{1,3\}) = 1$, $v(\bar{x}_1; \{2,3\}) = 4$, $v(\bar{x}_1; \{1\}) = \frac{1}{3}$, $v(\bar{x}_1; \{2\}) = \frac{1}{2}$, $v(\bar{x}_1; \{3\}) = \frac{1}{2}$. And the Shapley Value in the subgame $\Gamma(\bar{x}_1)$ is equal to

$$Sh(\bar{x}_1) = \left(\frac{34}{36}, \frac{91}{36}, \frac{91}{36}\right),$$

and we see that $Sh(x_0) \neq Sh(\bar{x}_1)$.

Time inconsistency of the Shapley Value



Consider now the subgame starting on cooperative trajectory from vertex $\bar{x}_2$. It can be easily seen that $v(\bar{x}_2; \{1,2,3\}) = 4$, $v(\bar{x}_2; \{1,2\}) = 1$, $v(\bar{x}_2; \{1,3\}) = 1$, $v(\bar{x}_2; \{2,3\}) = 4$, $v(\bar{x}_2; \{1\}) = \frac{1}{3}$, $v(\bar{x}_2; \{2\}) = \frac{1}{2}$, $v(\bar{x}_2; \{3\}) = \frac{1}{2}$. And the Shapley Value in the subgame $\Gamma(\bar{x}_2)$ is equal to

$$Sh(\bar{x}_1) = \left(\frac{21}{18}, \frac{21}{18}, \frac{21}{18} \right),$$

and we see that $Sh(\bar{x}_0) \neq Sh(\bar{x}_1) \neq Sh(\bar{x}_2)$. It is obvious that $Sh(\bar{x}_3) = (3,3,3)$.

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IPD for Shapley Value in this game can be easily calculated

$$Sh(0) = \left(\frac{44}{36},\ -\frac{22}{36},\ -\frac{22}{36}\right) + Sh(1),$$

$$Sh(1) = \left(-\frac{8}{36},\ \frac{49}{36},\ -\frac{41}{36}\right) + Sh(2),$$

$$Sh(2) = \left(-\frac{15}{18},\ -\frac{15}{18},\ \frac{30}{18}\right) + Sh(3),$$

$$Sh(3) = (3,\ 3,\ 3).$$

The strongly time consistency condition is more obligatory. We cannot even derive the formula like (5).